Current and Potential Distributions in Nonequilibrium MHD Plasmas at High Magnetic Field Strengths

LAJOS L. LENGYEL*
Max-Planck-Institut für Plasmaphysik, Garching bei München, Germany

The method of characteristics is applied to steady-state two-dimensional current and potential distributions in loss-free nonequilibrium MHD plasmas. The existence conditions of real characteristics are analyzed; the characteristic directions are defined and the corresponding compatibility relations are found. In an approximation corresponding to low ionization degree and moderate electron temperature elevation, analytic solutions are obtained for two particular cases: neutral collision and Coulomb collision dominated plasmas. The two families of characteristics are shown to coincide (neutral collisions) or nearly coincide (Coulomb collisions) with current streamlines and electric field lines. At magnetic field strengths above a critical value the discharge becomes striated even in ideal perturbation-free plasmas. The current flowing from electrode to electrode contracts to filaments of elevated current density. The solution indicates the existence of eddy currents in the system which consume a part of the applied (or induced) voltage, thus impairing the attainable electrical efficiencies.

Introduction

THE MHD power conversion experiments with pure and alkali metal seeded noble gases have raised a number of questions regarding the structure and behavior of discharges in nonequilibrium plasmas subject to the action of applied magnetic fields (see Refs. 1–6 and the studies cited therein). Some of the fundamental phenomena pertaining to current and potential distributions in such discharges can be explained on the basis of earlier analytical and numerical studies in which the nonlinear coupling between the current density and local plasma parameters was not taken into account (for example, Refs. 7–10). Quantitative data on the magnitude of these effects in nonequilibrium plasmas as well as information on the effects of finite rate ionization and recombination, velocity sweep, preionization, etc., on the field distributions in MHD channels and on the output characteristics of such devices could only be obtained from numerical studies allowing for the current dependence of the electron temperature, electron density, and the related plasma parameters.[11–16]

The computations involve numerical solution of nonlinear second-order partial differential equations obtained for the two-dimensional current streamline and/or electrostatic potential distributions from Ohm's generalized law and Maxwell's equations. The equations are usually linearized in the process of solution: only the basic variables (the components of the current density or electric field vectors expressed, respectively, by means of a stream and a potential function) appearing explicitly in the respective equations are treated as unknowns, while the coefficients [which are themselves functions of the current density or electric field strength, see Eqs. (6) and (7) of the next section] are computed and readjusted in an iterative manner. As a result of this approach, the highest-order derivatives of the unknown functions are not affected by the relative magnitudes of the coefficients appearing in the equations, and the equations thus linearized remain elliptic at any values of the physical parameters specified for the problem.

While at low magnetic field strengths the computations were found to converge rapidly and yield stable solutions,[12–14] convergence difficulties or unstable phenomena[15–18] were observed at magnetic field strengths exceeding a certain value. Systematic investigations of the original nonlinear equations[13,4] showed that the equations are only elliptic if the Hall parameter (magnetic field) does not exceed a critical value and become hyperbolic in the region above this value. As has been noted by Oliver and Mitchner,[13] the boundary between the elliptic and hyperbolic regions coincides with the stability criterion of electrothermal waves deducible from the same set of equations by means of perturbation analyses. Indeed, two-dimensional time-dependent computations carried out with the same equations have shown that while at low magnetic field strengths disturbances introduced into an initially uniform current field rapidly disappear from the system, at high magnetic field strengths they become amplified and lead to breakdown of the plasma into alternating layers of higher and lower current density, electron temperature, etc.[17,18] Only in a limited number of cases did these computations lead to reproducible asymptotic (quasi-stationary) field distributions other than final turbulent states[18] which are rather difficult to distinguish from computer-damped or smeared solutions.

The convergence difficulties inherent in these computations at high magnetic field strengths raise a question of a fundamental nature.

In the case of an elliptic boundary value problem, a disturbance introduced into the field at a boundary or an interior point is transferred from point to point until it affects the field distribution at all interior points, i.e., the "range of influence" of each point convers the whole domain considered. In the case of a hyperbolic problem, on the other hand, each boundary or interior point only has a limited range of influence. Disturbances introduced into the field at boundary or interior points as well as possible discontinuities in the derivatives of the independent variable can only propagate in well defined "characteristic" directions along the so called characteristic lines. All points outside the region bounded by the characteristics passing through a point (i.e., outside the range of influence of this point) remain unaffected by changes taking place at the point itself.

Hence treating a basically hyperbolic boundary value problem as an elliptic one means neglecting the possible discontinuities of the field derivatives and artifically extending the range of influence of each boundary or interior point to cover the whole domain. The validity of such an approach in general and its applicability to boundary value problems with discontinuous boundary conditions in particular are highly questionable.

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* Scientist; presently on leave to the Lorand Eötvös University of Sciences, Budapest, Hungary.

Recently, Glushkov et al.[4] have shown that in a particular case corresponding to a simplified plasma model the equations defining the two-dimensional current streamline distribution are reducible to a first-order partial differential equation which can be integrated at any magnetic field strength. The integral contains, however, a function that cannot be determined from the information provided by the statement of the problem.

The purpose of the present analysis is to explore field distributions in nonequilibrium MHD plasmas at high magnetic field strengths without manipulating the hyperbolic character of the basic equations involved. The applicability of the method of characteristics for computing steady-state two-dimensional electromagnetic field distributions is considered.

Characteristic Equations

Consideration is given to a two-dimensional nonequilibrium MHD plasma model (potassium seeded argon, $m_A \approx m_K$) describable in terms of the following equations:

$$\mathbf{J} + \beta(\mathbf{J} \times \hat{b}) = \sigma \mathbf{E} \tag{1}$$

$$\nabla \times \mathbf{E} = \nabla \cdot \mathbf{J} = 0 \tag{2}$$

$$\mathbf{J}.\mathbf{E} = n_e \nu_e \delta (3kT_e/2 - 3kTg/2) m_e/m_A \tag{3}$$

$$n_e = n_{e\text{Saha}}(T_e) \qquad \nu_e = \nu_e(T_e) \tag{4}$$

$$\sigma = n_e e^2/m_e \nu_e \qquad \beta = eB/m_e \nu_e \tag{5}$$

where $\hat{b}$ is a unit vector in the direction of the applied (uniform and constant) magnetic field: $\mathbf{B} = \hat{b}B_0 \| \hat{z}$. In the preceding equations, the usual notation has been used ($\mathbf{J}$ = current density, $\mathbf{E}$ = electric field intensity, β = Hall parameter, σ = electrical conductivity, n_e = electron density, T_e = electron temperature, ν_e = electron collision frequency, etc.). The plasma is at rest, steady-state conditions are assumed, the gradient of the electron pressure is neglected in Ohm's law. Only ohmic heating and elastic collisions are considered in the energy equation, the electron density is given by the Saha relation, and the collision frequency of the electrons is assumed to be a unique function of the electron temperature. The form of the $\nu_e = \nu_e(T_e)$ dependence is, however, left open. The analysis may be applied to an arbitrary noble gas alkali metal mixture by replacing the ratio ν_e/m_A in the energy equation by the sum

$$\sum_k \nu_{ek}/m_k$$

Variations in the direction of the magnetic field are neglected: $\partial/\partial z = 0$.

By means of Eqs. (2) one can eliminate either $\mathbf{E}$ or $\mathbf{J}$ from Eq. (1) thus obtaining two equivalent sets of vector equations defining the current and/or potential distributions in the plane perpendicular to the magnetic field direction $\hat{b}$:

$$\nabla \times \mathbf{J} + \mathbf{J} \times \nabla \ln\sigma - \hat{b}\beta \mathbf{J}\cdot\nabla \ln(\beta/\sigma) = 0, \nabla\cdot\mathbf{J} = 0 \tag{6}$$

and

$$\nabla\cdot\mathbf{E} + \mathbf{E}\cdot\nabla \ln\sigma + \beta(\mathbf{E} \times \nabla \ln\sigma)\cdot\hat{b} - [2\beta\mathbf{E} - (\beta^2 - 1)(\mathbf{E} \times \hat{b})]\cdot\beta\nabla \ln\beta/(1 + \beta^2) = 0 \tag{7}$$
$$\nabla \times \mathbf{E} = 0$$

Defining two logarithmic rate coefficients

$$\eta_s \equiv (d \ln n_e/d \ln T_e)_{\text{Saha}} = (3/2 + \varepsilon_i/kTe)(1 - \alpha)/(2 - \alpha)$$

and

$$\eta_\nu \equiv d \ln\nu_e/d \ln T_e \tag{8}$$

where $\alpha = n_e/(n_e + n_K)$ is the degree of ionization, the logarithmic derivatives appearing in Eqs. (6) and (7) can readily be expressed with the help of Eqs. (3) to (5):

$$\begin{aligned} d \ln\sigma &= (\eta_s - \eta_\nu)d \ln T_e = (1 - \eta)\eta_s d \ln T_e \\ d \ln\beta &= -\eta_\nu d \ln T_e \\ d \ln T_e &= (\eta_s + \Delta T_e/2)^{-1} d \ln J = [\eta_\nu/(1 + \beta^2) + \Delta T_e/2]^{-1} d \ln E \end{aligned} \tag{9}$$

where $\eta = \eta_\nu/\eta_s$, and $\Delta T_e = T_e/(T_e - T_g)$ is a measure of the electron temperature elevation.

In view of Eq. (2), a potential function ϕ and a current stream function γ can be introduced by means of the following relations:

$$\mathbf{E} = -\nabla\phi, \ \mathbf{J} = \nabla \times \boldsymbol{\gamma}, \ \boldsymbol{\gamma} = (0,0,\gamma) \tag{10}$$

The nonlinear second-order partial differential equations that define the potential and current distributions are obtained by substituting Eqs. (9) and (10) into Eqs. (6) and (7) and can be written in the following form:

$$A_f \partial^2 f/\partial x^2 + B_f \partial^2 f/\partial x \partial y + C_f \partial^2 f/\partial y^2 = 0 \tag{11}$$

where $f \equiv \phi$ and $f \equiv \gamma$ for the potential and the current streamline distributions, respectively. The coefficients A_f, B_f, and C_f are themselves functions of the indepentend variable. Equation (11) is supplemented by the boundary conditions $E_{\tan} = 0$ at electrode surfaces and $J_{\text{norm}} = 0$ at insulator surfaces.

The characteristic equation and the respective compatibility relation for either of the two field distributions (ϕ,γ) are given by the pair of zero matrices

$$\begin{vmatrix} A_f & B_f & C_f \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0, \quad \begin{vmatrix} A_f & 0 & C_f \\ dx & df_x & 0 \\ 0 & df_y & dy \end{vmatrix} = 0 \tag{12}$$

The indices x and y appearing at the function "f" (i.e., at ϕ and γ) denote derivatives with respect to these variables. At all other quantities they denote projections on the respective coordinate axes. The first matrix yields a first-order differential equation defining two families of characteristics ("+" and "−") for each of the field distributions

$$\left.\frac{dy}{dx}\right)_\phi^\pm = \frac{1}{2}\frac{b[p + \beta + p(1 - \beta p)] - 2b_\nu(p + \beta)(1 - \beta p)/(1 + \beta^2) \pm (1 + p^2)\mathfrak{D}_\phi}{b(1 - \beta p) + b_\nu[(\beta + p)^2 - (1 + p^2)]/(1 + \beta^2) + 1 + p^2} \tag{13}$$

with

$$\mathfrak{D}_\phi^2 = \beta^2[b - 2b_\nu/(1 + \beta^2)]^2 - 4[b + 1 + b_\nu(\beta^2 - 1)/(\beta^2 + 1)]$$

and

$$\left.\frac{dy}{dx}\right)_\gamma^\pm = \frac{1}{2}\frac{\beta b_s(q^2 - 1) + 2b_s(1 - \eta)q \pm (1 + q^2)\mathfrak{D}_\gamma}{\beta b_s q - (1 - \eta)b_s q^2 + 1 + q^2} \tag{14}$$

with

$$\mathfrak{D}_\gamma^2 = \beta^2 b_s^2 - 4[1 - (1 - \eta)b_s]$$

In the preceding expressions, the following notation has been used:

$$p = E_y/E_x = (\partial\phi/\partial y)/(\partial\phi/\partial x) = (q - \beta)/(1 + q\beta) \tag{15a}$$

$$q = J_y/J_x = -(\partial\gamma/\partial x)/(\partial\gamma/\partial y) = (p + \beta)/(1 - p\beta) \tag{15b}$$

$$b_\nu = \eta_\nu/[\eta_\nu/(1 + \beta^2) + \Delta T_e/2] \tag{15c}$$

$$b_s = \eta_s/(\eta_s + \Delta T_e/2) \tag{15d}$$

$$b = \eta_s b_\nu/\eta_\nu \quad (15e)$$

The second matrix (compatibility relation) yields expressions for the variation of the current density and electric field components along the respective characteristics. Because of the homogeneity of the basic equations, the two families of characteristics in the physical (x,y) and "hodograph" (E_x, E_y or J_x, J_y) planes are found to be reciprocally orthogonal

$$(d\phi_y/d\phi_x)^{\pm}(dy/dx)_\phi^{\mp} = (dE_y/dE_x)^{\pm}(dy/dx)^{\mp} = -1 \quad (16)$$

$$-(d\gamma_y/d\gamma_x)^{\pm}(dy/dx)_\gamma^{\mp} = (dJ_x/dJ_y)^{\pm}(dy/dx)^{\mp} = 1 \quad (17)$$

The existence of real characteristics depends upon the magnitude of the quantities appearing in $\mathfrak{D}_\phi$ and $\mathfrak{D}_\gamma$. Since the current and potential distributions are not independent of each other, $\mathfrak{D}_\phi$ and $\mathfrak{D}_\gamma$ must be linearly dependent. As can readily be shown, $b_s\mathfrak{D}_\phi = b\mathfrak{D}_\gamma$.

A check of the expressions for $\mathfrak{D}_\phi$ and $\mathfrak{D}_\gamma$ reveals that, within the framework of the present approximation, no real characteristics exist in equilibrium plasmas ($\Delta T_e \to \infty$, $b \to b_s \to b_\nu \to 0$). In the absence of magnetic fields ($\beta = 0$), or in the case of a fully ionized plasma ($\alpha \to 1$, $\eta_s \to b \to b_s \to 0$) real characteristics exist if $\eta_\nu < -\Delta T_e/2$, i.e., if the electron collision frequency decreases with increasing electron temperature and the rate of decrease satisfies this inequality. In the presence of a magnetic field, real characteristics exist if the Hall parameter exceeds a critical value given by $\beta_{\text{crit}} = 2[(\eta_s + \Delta T_e/2)(\eta_\nu + \Delta T_e/2)]^{1/2}/\eta_s$. Hence

$$\mathfrak{D}_\phi/b = \mathfrak{D}_\gamma/b_s = (\beta^2 - \beta_{\text{crit}}^2)^{1/2} \quad (18)$$

It is of interest to note that the existence condition for real characteristics coincides with the stability limit of the basic equations with respect to small perturbations. Indeed, a linear perturbation analysis applied to these equations (the energy equation is taken in its time-dependent form) yields a maximum exponential growth rate ω_i given by the following expression:

$$\tau^*\omega_i = [\beta^2\eta_s^2 + (\eta_s - \eta_\nu)^2]^{1/2} - (\eta_s + \eta_\nu + \Delta T_e)$$

where

$$\tau^* = (3kT_{e0}\eta_{e0}\sigma_0/2J_0^2)[1 + (1 + 2\varepsilon_i/3kT_{e0})\eta_{s0}]$$

is a characteristic growth time corresponding to Saha conditions. As can readily be seen, the cirtical Hall parameter corresponding to neutral stability ($\omega_i = 0$) is identical to the one representing the existence condition of real characteristics. Hence in this approximation at high magnetic field strengths steady-state field distributions may only exist under ideal (perturbation-free) conditions.

We shall now consider in detail two particular cases corresponding to neutral and Coulomb collision dominated plasmas. It shall be assumed in both cases that the value of the coefficient $\Delta Te/2\eta_s$ is small compared to unity (low degree of ionization, etc.). Although this condition allows considerable simplification of the equations involved, it has some undesirable consequences. As can be seen from Eqs. (15) and (9), if $\Delta Te/2\eta_s \ll 1$, $b_s \to 1$, and $d\ln n_e \approx d\ln J$, i.e., the electron density vanishes in regions of zero current density. Such a mathematical abstraction warrants special care in interpreting results which involve locally vanishing current densities.

Because of computational convenience, we shall work primarily with the equations derived for the current streamline distribution. A Faraday generator geometry with plane electrodes shall be considered.

A. Neutral Collision Dominated Plasma

Since in this case the collision frequency is practically independent of the electron temperature, $\eta_\nu \approx 0$, $\eta \to 0$, $b_\nu \to 0$, and $b \gg 1$. Furthermore, since $b_s \to 1$, $d\ln\sigma \approx d\ln J$ [see Eq. (9)], and Eq. (14) reduces to

$$(dy/dx)_\gamma^{\pm} = [\beta(q^2 - 1) + 2q \pm \beta(q^2 + 1)]/2(1 + \beta q) \quad (19)$$

Using the reciprocal orthogonality condition and Ohm's law we obtain the rather simple relations

$$(dy/dx)^+ = q,\ (dJ_y/dJ_x)^+ = p$$
$$(dy/dx)^- = p,\ (dJ_y/dJ_x)^- = q \quad (20)$$

where the subscript "γ" has been omitted from the slope of the physical characteristics. [The same relations may also be obtained from Eq. (13).]

The variation of the electric field components along the characteristics can be determined from Ohm's law and the last of Eq. (9):

$$(1 + pdE_y/dE_x)[\beta + p - (1 - \beta p)dJ_y/dJ_x] \times (2\eta_s/\Delta T_e)/(1 + p^2) + (1 + \beta dJ_y/dJ_x)dE_y/dE_x + \beta - dJ_y/dJ_x = 0 \quad (21)$$

Equations (20) and (21) are supplemented by the relations

$$dp = (dE_y/dE_x - p)dE_x/E_x \quad (22a)$$

$$dq = (dJ_y/dJ_x - q)dJ_x/J_x \quad (22b)$$

As can be seen from Eq. (20), in this approximation the two families of characteristics coincide with the current streamlines and the electric field lines, respectively. Disturbances caused, for example, by discontinuities in the boundary conditions (electrode edges) can only propagate along current streamlines or electric field lines into the plasma.

Equations (20–22) yield the following expressions for the variation of the electric field intensity and current density along the positive and negative characteristics:

$$d\ln J)^+ = -d\theta_q/\beta,\ d\ln E)^+ = -d\theta_p/\alpha^*\beta$$
$$d\ln J)^- = 0,\ d\ln E)^- = 0 \quad (23)$$

where $\theta_q \equiv \arctan q$, $\theta_p \equiv \arctan p$, $\theta_q = \theta_p + \arctan\beta$ and $\alpha^* \equiv 1 + 2\eta_s/\Delta T_e \approx 2\eta_s/\Delta T_e$. Since $\beta = $ const in this approximation, the preceding expression can readily be integrated:

$$J)^+ = \text{const}\exp(-\theta_q/\beta),\ E)^+ = \text{const}\exp(-\theta_p/\alpha^*\beta)$$
$$J)^- = \text{const},\ E)^- = \text{const} \quad (24)$$

The variations of J_x, J_y, E_x, and E_y as functions of the streamline or electric field directions can readily be computed using Eq. (24) and the definitions of q and p. The constants of integration can be found from the boundary conditions specified at, for example, electrode surfaces.

As can be seen, both **E** and **J** remain constant in regions where p and q are constant and vary elsewhere according to the logarithmic spiral law. Straight streamlines can only be parallel lines, confocal (converging or diverging) rays are not admitted by this solution. Following a streamline, a turn in the positive (counter-clockwise) direction reduces the current density, and vice versa.

As can be seen from Eq. (22), both p and q remain constant along the negative characteristics, which are electric field lines. Hence the electric field lines must be straight lines normal to equipotential surfaces and emerging from or terminating on nonvanishing charge concentrations. The characteristics are parallel lines at plane electrode surfaces and concentric rays at point charges. Thus the centered waves arrived at in Ref. 4 are in fact electric field lines belonging to the family of negative characteristics considered here. The equation of the current streamlines can be written at once by recalling that the separation distance between the streamlines is inversely proportional to the current density: $r = \text{const}\exp(\theta/\beta)$, i.e., the streamlines are represented by exponential spirals. Here $r = [(x - x_c)^2 + (y - y_c)^2]^{1/2}$, $\theta = \arctan[(y - y_c) - x - x_c)]$, and subscript c refers to the location of the spiral

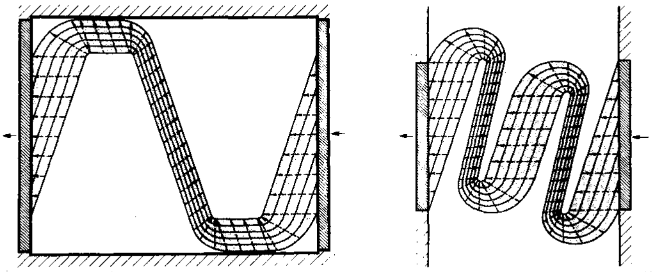

Fig. 1 Hyperbolic field distributions in neutral collision dominated plasmas with discontinuous boundary conditions. Solid lines: current stream lines. Broken lines: electric field lines. a) Finite electrode segmentation; b) lateral restriction.

center. The same expression can also be obtained formally by integrating the first of Eqs. (20).

Current and electric field distributions arrived at on the basis of the above solution are shown in Fig. 1. The main features of these distributions are as follows:

a) The current flowing from electrode to electrode does not fill the whole space available: it contracts to filaments of elevated current density. The effect of finite electrode segmentation, i.e., the presence of electrode edges is "sensed" by the current streamlines only when they reach the negative characteristics (electric field lines) emerging from the points of discontinuity (see Fig. 1a). Charge concentrations appear in the field which "rotate" the electric field lines and thus turn the current streamlines. The solution admits multiple turns. In an ideally homogeneous plasma the number of turns and of associated high current density filaments in limited to one (minimum resistance principle). The maximum possible number of turns (filaments) is given by $(H/C)\beta/(1 + \exp\pi/\beta)$, where C is the electrode width. The current distribution along the electrode surface is arbitrary.

b) If the current flow is restricted in the lateral direction (insulator walls, see Fig. 1b), the current constricts at the electrode and part of the electrode surface becomes idle. The disturbance caused by the presence of the insulator wall propagates along the positive characteristics (current lines) to the electrode surface. The coverage ratio of the electrode surface is a function of the Hall parameter.

c) The direction of the high current density layers (modes) admitted by this solution depends strongly on the number of layers actually existing in the system and the electrode geometry ($\beta H/C$ ratio). Considering a single mode, if $\beta H \gg C$, the direction of the current filament almost coincides with that of the mean current. As the distance between the anode and cathode decreases, the current filament turns gradually in the clockwise direction until it becomes nearly antiparallel to the current direction at the electrodes.

Let us finally examine to what degree the field distributions thus obtained can be considered as limit cases inherent in neutral collision dominated plasmas. Should they represent a true limit, small deviations in any of the assumptions made may not lead to qualitative changes in the field distributions. Assume, for example, that the equilibrium conductivity is small, but different from zero. Since the electric field lines cannot terminate at the boundaries separating high conductivity and low conductivity regions (curl $\mathbf{E} = 0$), it follows from Ohm's law that the entire space available must be filled with current. The present solution does not admit diverging current stream lines other than spiral ones, hence the current in the low σ region cannot come from the electrodes. Current stream tubes closed on themselves (eddy currents) are not admitted either by this solution. Hence, the solution in its present form loses its meaning as soon as one attempts to extend it to the whole domain.

This apparent contradiction can be resolved by examining once more the basic assumptions involved. As a result of the neutral collision approximation and the $\Delta Te/2\eta_e \ll 1$ assumption, β_{crit} approaches zero, i.e., the field distributions are hyperbolic at any magnetic field strength and current density values. On the other hand, in regions where $\sigma \approx 0$, ΔTe and thus β_{crit} approach infinity, i.e., the distributions remain elliptic irrespective of the magnetic field strength. Hence, regions with $\sigma \approx 0$ are not and cannot be considered as parts of the present (hyperbolic) solution.

A plausible approach therefore seems to be to assume that in the low conductivity regions the plasma is filled with eddy currents whose distribution can be found by solving an elliptic boundary value problem defined by the complete set of equations (dissipative effects included) and the boundary conditions E_{tan} = const and $J_{norm} = 0$ at the high conductivity low conductivity interfaces (represented by streamlines) and $J_{norm} = 0$ at insulator surfaces. A possible discharge pattern satisfying the above conditions is shown in Fig. 2 for periodic boundary conditions. The eddy currents represent pure ohmic losses: part of the applied voltage (or EMF) is wasted on internally circulating currents. The return loops of the eddy currents may again form layers of elevated current density which have no connection with the electrodes and are independent of the electrode segmentation. The possibility of several such layers between two neighboring electrode pairs is not excluded.

B. Coulomb Collision Dominated Plasma

It shall be assumed in this case that the electron collision frequency is directly proportional to the electron number density, i.e., $d \ln\nu_e/d \ln n_e \approx 1$, $\eta_\nu \approx \eta_s$, and $\eta \approx 1$. With $b_s \to 1$ it follows from Eq. (9) that $d \ln J = d \ln n_e = -d \ln\beta$ and from Eq. (14) that $\beta_{crit} = 2$.

The equation of characteristics obtained for the current distribution from Eq. (14) can be written in the following form:

$$(dy/dx)^{\pm} = \beta/2[q^2 - 1 \pm C_\beta(q^2 + 1)]/(q^2 + \beta q + 1) \quad (25)$$

where

$$C_\beta = (1 - 4/\beta^2)^{1/2}$$

or, using the reciprocal orthogonality condition

$$(dy/dx)^+ = (dJ_y/dJ_x)^- = [\beta q(1 + C_\beta)/2 - 1]/[q + \beta(1 + C_\beta)/2] \quad (26)$$

$$(dy/dx)^- = (dJ_y/dJ_x)^+ = [\beta q(1 - C_\beta)/2 - 1]/[q + \beta(1 - C_\beta)/2]$$

The variation of the electric field components along the characteristics is given in terms of the current density variation by means of the following relation (obtained from Ohm's law):

$$(1 + p dE_y/dE_x)[q^2 + 1 - \beta(q - dJ_y/dJ_x)](1 + \beta^2)/(1 + p^2) - (1 + \beta q)(1 + q dJ_y/dJ_x) = 0 \quad (27)$$

Unlike in neutral collision dominated plasmas, here the characteristics do not coincide with current streamlines nor with electric field lines. The angles between the characteristics and the respective field lines are given by $\tan[\measuredangle \mathbf{J},C^+] = [\beta(1 + C_\beta)/2]^{-1}$ for the positve characteristics, and $\tan[\measuredangle \mathbf{E},C^-] = -[2 + \beta(1 + C_\beta)]^{-1}$ for the negative characteristics. Hence with increasing Hall parameter values the characteristic directions approach asymptotically the local current and electric field directions, respectively.

Equations (26) and (27) supplemented by Eq. (22) yield the following expressions for the variation of the electric field and current density components along the characteristics:

$$d\ \ln J)^+ = -\beta/2(1 - C_\beta)d\theta \quad (28a)$$

$$d\ \ln J)^- = -\beta/2(1 + C_\beta)d\theta \quad (28b)$$

$$d\ \ln E)^+ = -\beta/2[(1 - C_\beta)^2/(3 - C_\beta)]d\theta \quad (28c)$$

$$d\ \ln E)^- = -\beta/2[(1 + C_\beta)^2/(3 + C_\beta)]d\theta \quad (28d)$$

where $\theta = \arctan(J_y/J_x)$ represents the direction of the current density vector. These expression can be integrated numerically or, by taking into account the relation between the current density and the Hall parameter inherent in this approximation, also analytically. The final relations thus obtained can be written in parametric form with the Hall parameter as independent variable:

$$(\theta - \theta_0)^\pm = \beta/2[1 \pm (1 - 4/\beta^2)^{1/2}] \pm \arcsin(2/\beta) \quad (29)$$

$$J)^\pm = \text{const}/\beta, \quad E)^\pm = \text{const}(1 + 1/\beta^2)^{1/2}$$

where the constants of integration are given by the initial values specified for θ, J, and E, respectively. The hodograph solution obtained for the current density is shown in Fig. 3 for three different initial Hall parameter values. Solid and broken lines denote variations along positive and negative characteristics, respectively. Obviously, with $\theta_0^+ = \theta_0^-$ the shift between θ^+ and θ^- corresponding to any particular Hall parameter value is given by $\theta^+ - \theta^- = \beta(1 - 4/\beta^2)^{1/2} + 2\arctan(2/\beta)$. However, for the purpose of better visualization, the initial angles θ_0^+ and θ_0^- have been selected in such a manner that all initial points coincide (point $J/J_0 = 1$ in Fig. 3).

Examination of these results leads to the following conclusions:

a) Following a characteristic (positive or negative) a turn of the current streamlines in the counterclockwise direction reduces the current density, and vice versa. At high Hall parameter values [$C_\beta \rightarrow 1$, see Eq. (28)], variations along positive characteristics are much weaker than those along negative ones.

b) The variation of the current direction along the positive characteristics is unbounded [see Eq. (29) and Fig. 3]. The magnitude of the current density vector varies along positive characteristics in a manner similar to the neutral collision dominated case: the contraction of the spiral radius over a given angle interval is approximately given by $\exp(-\Delta\theta/\langle\beta\rangle)$, where the Hall parameter is averaged over the respective angle range.

c) The variation of the current direction along the negative characteristics is confined to a relatively narrow range of admissible angle values [see Eq. (29) and Fig. 3]:

$$1 - \pi/2 \leq (\theta - \theta_0) < 0 \quad (30)$$

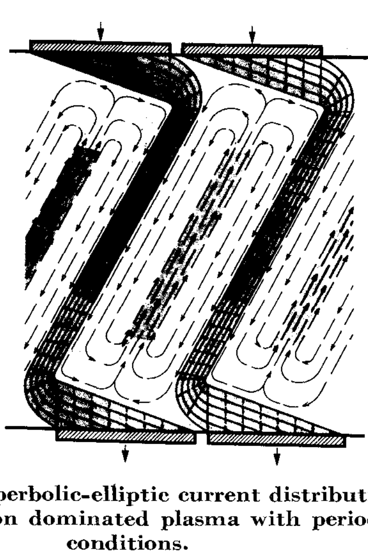

Fig. 2 Hyperbolic-elliptic current distribution in a neutral collision dominated plasma with periodic boundary conditions.

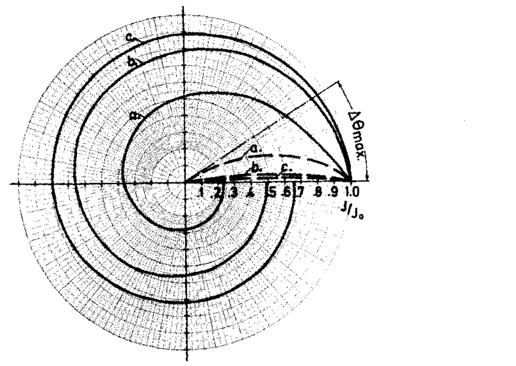

Fig. 3 Hodograph solution for a Coulomb collision dominated plasma. Solid lines: variation along positive characteristics. Broken lines: variation along negative characteristics. a) $\beta_0 = 2$; b) $\beta_0 = 6$; c) $\beta_0 = 12$.

whereby the lower and upper limits correspond to $\beta = 2$ and $\beta = \infty$, respectively. The higher the operating Hall parameter range, the smaller is the admissible angle variation. A relatively small change of the current direction causes a very sharp increase (or decrease) of the current density along the negative characteristics. Since at high β values the characteristics are nearly parallel to the electric field lines and thus perpendicular to the current streamlines, an increase (or decrease) of the current density along the C^- lines is equivalent to a sharp contraction (or expansion) of the current streamlines.

d) No solution exists outside the range of current direction variation admitted by Eq. (30). Hence only a limited number of special geometrical configurations (staggered electrodes, etc.) can be satisfied by this solution.

The last result, i.e., the restriction of the admissible current variation to a small fraction of the hodograph plane, has a sound physical reason: the range of admissible θ^- values is defined in terms of the existence conditions of real characteristics. Indeed, no real characteristics exist if $\beta < \beta_{crit} = 2$, or if the current density is insufficient to produce nonequilibrium ionization (in reality the condition of dominating Coulomb collisions is violated long before this minimum current density is reached). The lower current density limit—should it be at all possible to define it in terms of a unique value—would be represented in the hohograph plane by a circle, drawn around the center of the coordinate system. As soon as the limiting Hall parameter values (current densities) are reached, the solution ceases to be hyperbolic.

Let us finally consider, wheather for $\beta < \beta_{crit}$ (the plasma is still Coulomb collision dominated) an analytical continuation of the solution into the elliptic domain—in a manner analogous to the neutral collision dominated case—would still be possible. The corresponding elliptic boundary value problem with a given current flux along a part of the boundary (the constricted current filament entering the field) would yield expanding current streamlines which would tend to fill the whole space available. A reduction of the current density would, however, increase the local Hall parameter above its critic value, thus shifting the current distribution back into the hyperbolic region. Hence a transition from the hyperbolic to the elliptic domain is impossible. On the other hand, the possibility of a purely parabolic solution ($\beta = \beta_{crit} = $ const) is ruled out by the fact that the solution obtained in this approximation:

$$(dy/dx)^\pm = (q - 1)/(q + 1), \quad \tan[\measuredangle J, C] = 1 \quad (31)$$

$$d\ \ln J)_c = d\theta, \quad d\ \ln J)_q = (2)^{1/2}/2d\theta$$

implies a current density varying exponentially with the current direction (subscripts c and q denote variations along characteristics and current streamlines, respectively). Since

$\beta \propto 1/J$, the condition $\beta = \beta_{crit}$ = const cannot be fulfilled by this solution. Hence, within the present approximation, no solution exists for boundary conditions that require a turn of the current streamlines by an angle greater than $\Delta\theta_{max}$ (Fig. 3).

At high magnetic field strengths and for boundary conditions that comply with the above restriction, the current distributions in Coulomb collision dominated plasmas are similar to those obtained for the neutral collision dominated case, except the constriction and expansion of the current filaments is far more pronounced here than in the former case. The limited nature of this solution is a result of the mathematical abstraction used; of the assumptions $\Delta Te/2\eta_s \ll 1$ and $d \ln J = d \ln n_e = -d \ln \beta$ in particular. The removal of these assumptions would, however, require a numerical solution of the hodograph equations and a step-by-step (numerical) build-up of the characteristic network in the physical plane. Such a procedure is equivalent in volume to the numerical solution of the original set of equations and exceeds the scope of the present analysis.

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